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The Phase Transition in the Discrete Gaussian Chain with $1/r^2$ Interaction Energy

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We exhibit a phase transition from a rough high-temperature phase to a rigid (localized) low-temperature phase in the discrete Gaussian chain with $1/r^2$ interaction energy. This transition is related to a localization transition in the ground state for a quantum mechanical particle in a one-dimensional periodic potential, coupled to quantum $1/f$ noise.

KEY WORDS: Phase transition; $1/r^2$ discrete Gaussian; localization; quantum $1/f$ noise; Peierls argument; correlation inequalities; energy–entropy argument.

1. INTRODUCTION

1.1. The Main Results

In this paper we rigorously establish a transition from a rough, delocalized high-temperature phase to a rigid, localized low-temperature phase in the discrete Gaussian chain with $1/r^2$ interaction energy. The Hamiltonian of the discrete Gaussian chain is chosen to be

$$H^{\mathrm{DG}}(n)=\frac{1}{2}\sum_{i,j} g(i-j)(n_i-n_j)^2 \tag{1.1}$$

where i and j range over $\mathbb{Z}$, $n_i \in \mathbb{Z}$ is a height variable (e.g., the height of an interface over i, or the difference of the position of a particle to its equilibrium position), and $g(i-j)$ is a coupling function. We assume that

$$g(r)\sim r^{-\alpha}, \qquad \alpha=2, \qquad \text{as} \quad r\to\infty \tag{1.2}$$

This paper is dedicated to J. L. Lebowitz on the occasion of his 60th birthday.

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For $\alpha > 2$, the system with Hamiltonian (1.1) is expected to always be in the rough phase, i.e., there is a constant C_β, with $C_\beta \sim 1/\beta$, for β small, such that

$$\langle (n_i - n_j)^2 \rangle_\beta \sim C_\beta \, |i-j|^\gamma, \qquad \gamma \equiv \min(\alpha - 2, 1) \tag{1.3}$$

as $|i-j| \to \infty$, while for $\alpha < 2$, a rigid phase with (C'_β a β-dependent constant)

$$\langle n_i^2 \rangle_\beta \lesssim C'_\beta \tag{1.4}$$

appears to persist, for all β. In (1.3) and (1.4), $\langle (\cdot) \rangle_\beta$ denotes the equilibrium state determined by H^{DG} with 0-boundary conditions at inverse temperature β. Behavior (1.3) is related to behavior (1.4) by a duality transformation, so that (for suitably chosen boundary conditions) it is enough to prove (1.3), say. This is not a particularly easy task if one wants to establish (1.3) for *all* values of β. [A Peierls argument—see Section 3—can be used to prove (1.4), for large enough values of β. We expect that (1.4) could be proven directly, for smaller values of β, by using a finite, but β-dependent sequence of renormalization transformations, followed by a Peierls argument.]

In order to explain why we do not expect to observe a phase transition in the discrete Gaussian chain for interactions with $g(r) \sim r^{-\alpha}$, $\alpha \neq 2$, while, for $\alpha = 2$, a transition will be proven to exist, we shall now discuss a heuristic *energy–entropy argument*. We first discuss the models with $\alpha > 2$. (The models with $\alpha < 2$ can be understood in terms of the models with $\alpha > 2$ by using a duality transformation.[(3)]) We shall first estimate how much energy needs to be paid for the interface described by the discrete Gaussian chain, with n_j interpreted as the height of the interface at site j, to reach a "macroscopic" height. From the form (1.1) of the Hamiltonian we see that configurations of height variables of comparatively small energy are those which exhibit only unit jumps. More precisely, for the system confined to the interval $\Lambda = [-L, L] \subset \mathbb{Z}$ with zero boundary conditions, the relevant configurations can be described as follows: A configuration n of height variables is said to have unit jumps only iff there is a partition $\{I_\lambda\}_{\lambda = -K, \ldots, K}$ of Λ into $2K+1$ disjoint intervals with the property that

$$n_i = N_\lambda, \qquad \text{for all} \quad i \in I_\lambda \tag{1.5}$$

for integers N_λ such that $|N_{\pm K}| \leqq 1$, and

$$|N_\lambda - N_{\lambda \pm 1}| = 1 \tag{1.6}$$

Such "staircase" configurations $n(\{I_\lambda\})$ can reach a maximal height K. We shall now choose K, the integers N_λ, and the lengths $|I_\lambda|$ of the intervals I_λ, for $\lambda = -K, -K+1, ..., K$. We set

$$|I_\lambda| = a\,|\pi(\lambda)|^{2/\varepsilon} \tag{1.7}$$

where $\varepsilon := \alpha - 2$, π is an arbitrary permutation of $\{-K, -K+1, ..., 0, ..., K\}$, and $a > 0$ is some constant. Since

$$\sum_{\lambda=-K}^{K} |I_\lambda| = 2L$$

it follows from (1.7) that

$$K = O(L^{\varepsilon/(2+\varepsilon)}) \quad [\approx O(L^{\varepsilon/2}), \text{ for } \varepsilon \text{ small}] \tag{1.8}$$

Finally, we set

$$N_\lambda = K - |\lambda| \tag{1.9}$$

Thanks to the rapid growth of the lengths of the intervals $I_{\pi^{-1}(\lambda)}$ in $|\lambda|$, prescribed in (1.7), the *energy* $H(n(\{I_\lambda\}))$ of a configuration $n(\{I_\lambda\})$ satisfying (1.7)–(1.9) turns out to be bounded from *above* by a constant times the sum of *interaction energies of adjacent intervals* which, by (1.6) and (1.1), is proportional to K. Hence

$$H(n(\{I_\lambda\})) \leqslant bK \sim O(L^{\varepsilon/2}) \tag{1.10}$$

Clearly, the *entropy* of the set of configurations $n(\{I_\lambda\})$ satisfying (1.7)–(1.9) behaves like

$$S^{(K)} \sim O(K \ln K) \tag{1.11}$$

Thus, entropy dominates energy at arbitrary temperatures, and we conclude that the maximal height $n_{\max}$ of the interface behaves like

$$n_{\max} = O(L^{\varepsilon/(2+\varepsilon)}) \sim O(L^{(\alpha-2)/2}) \tag{1.12}$$

for $\alpha - 2$ small, at arbitrary temperature. It is well known that, for $\alpha > 3$, $n_{\max} = O(\sqrt{L})$, at all temperatures, as follows from a central-limit argument. We believe that these bounds could be made rigorous with some hard work.

Let us now turn to the case where $\alpha < 2$. It follows from (1.12) by duality that the interface is *rigid*, at arbitrary temperatures. This can also be understood directly: We give a heuristic estimate of the probability that the height n_0 at the origin is $|n_0| = K$, for some large, positive integer K.

For this event to occur, the origin must be in the interior of a configuration of jumps. Choosing these jumps to be bounded, there will be $O(K)$ jumps surrounding 0. The energy of this class of configurations grows at least like $K^{3-\alpha}$, while the entropy behaves like $S^{(K)} = O(K \ln K)$. Thus, the probability p_K of the event that $n_0 = K$ appears to behave like

$$p_K \leqq \exp(-\beta c K^{3-\alpha} + dK \ln K) \tag{1.13}$$

for some finite, positive constants c and d. We conclude from (1.13) that, for $\alpha < 2$,

$$p_K \to 0 \qquad \text{as} \quad K \to \infty \tag{1.14}$$

for arbitrary $\beta > 0$; hence the interface is always rigid.

When $\alpha = 2$, the behavior of the energy is given by $O(K \ln K)$, and we obtain

$$p_K \sim \exp[-(\beta c - d) K \ln K] \tag{1.15}$$

We thus expect that there is a phase transition from a rigid to a rough interface at some finite, positive value of β. This will be proven in Sections 3 and 4.

Let us remark that the arguments sketched above work for zero and for Dirichlet boundary conditions, with the same conclusions.

It is well known that for $\alpha > 2$ the corresponding *Ising model* has no phase transition. For the Ising model with $1/r^2$ interaction energy, the existence of a phase transition has been proven in ref. 1. The existence of a phase transition in the Ising model with $1 < \alpha < 2$ is an older result proven in ref. 6. Why does that not suggest that there is a phase transition in the corresponding discrete Gaussian chain, for $1 < \alpha < 2$? The reason is as follows: In order to find the behavior (1.13) for p_K, we must make sure that the distance between consecutive jumps satisfies some growth condition (linear growth). This is because, otherwise, the interaction energy between distant intervals would be too large, due to the factors $(n_i - n_j)^2$ appearing in the interaction energy. Without any growth conditions on the distances between consecutive jumps, the height variables n_j will therefore be constrained to remain close to 0. Thus, while jumps may become abundant at high temperatures, as they do in the Ising model, jumps to *large values* of $|n_0|$ remain unlikely, at arbitrary temperatures.

We now turn to a summary of our results for the discrete Gaussian chain with $\alpha = 2$. Our main result is that, for this model,

$$\langle (n_i - n_j)^2 \rangle_\beta \gtrsim \text{const}_\beta \cdot \log |i - j| \tag{1.16}$$

as $|i-j| \to \infty$, provided β is small enough, while

$$\langle n_i^2 \rangle_\beta \lesssim \text{const}'_\beta \tag{1.17}$$

if β is large enough.

Our method of proof is as follows: Kjaer and Hilhorst[3] have shown that if

$$g(r) \equiv g^*(r) = (r^2 - 1/4)^{-1} \tag{1.18}$$

then the model is self-dual at $\beta = 1$, and

$$\langle (n_i - n_j)^2 \rangle_{\beta=1} \simeq \frac{1}{2\pi^2} \log |i-j| \tag{1.19}$$

as $|i-j| \to \infty$. From this one can deduce (1.16) for g as in (1.18) and $\beta < 1$ with the help of correlation inequalities.[4] The inequalities in ref. 4 also permit us to extend (1.16) to coupling functions $g(r)$, with the property that

$$\hat{g}(k) \leqslant A\hat{g}^*(k) \tag{1.20}$$

for all $\beta < A^{-1}$; see Section 4.

The proof of (1.17) for large values of β is more difficult. It is based on an extension of the Peierls argument developed in ref. 1 for the $1/r^2$ Ising chain; see Section 3.

Inequalities (1.16) and (1.17) clearly demonstrate the existence of a roughening transition in our model, as β is lowered. This is of some interest in view of the role one-dimensional spin systems with $1/r^2$ interaction energy have played in the development of renormalization group techniques.

Our result (1.17) has an interesting consequence: Consider the correlation

$$G_\beta(i-j) \equiv \exp[-2\pi^2 \langle (n_i - n_j)^2 \rangle_\beta]$$

and define an exponent $\eta(\beta)$ by

$$G_\beta(r) \sim r^{1-\eta(\beta)} \tag{1.21}$$

Then, for $g = g^*$, defined in (1.18), we have the relation

$$\beta\eta(\beta) + \beta^{-1}\eta(\beta^{-1}) = 2 + \beta + \beta^{-1} \tag{1.22}$$

which follows by duality, as noted in ref. 3. Now, suppose that (1.17) holds for all $\beta > \beta_0 \geqslant 1$. Then

$$\eta(\beta) = 1 \qquad \text{for} \quad \beta > \beta_0$$

and (1.22) implies that, for $\beta < \beta_0$,

$$\eta(\beta) = 1 + 2/\beta \tag{1.23}$$

which coincides with the value of the corresponding exponent in the ordinary Gaussian model ($n_i \in \mathbb{R}$, for all i) with inverse covariance determined by g^*. The finiteness of β_0 follows from our results in Sections 3 and 4. It is an interesting open problem to prove or disprove that $\beta_0 = 1$.

For $g \neq g^*$, we can deduce upper and lower bounds on $\eta(\beta)$ from (1.23) with the help of the correlation inequalities in ref. 4.

1.2. Connection between the Discrete Gaussian Chain and the Quantum Mechanics of a Particle in a Periodic Potential, Coupled to Quantum Mechanical 1/*f* Noise

We consider a quantum mechanical particle moving on the real line under the influence of an external potential V and coupled to one component A_1 of a slowly varying gauge field. The position variable of the particle is denoted by x, its momentum by p. The one-particle Hamiltonian H_p is given by

$$H_p = \frac{1}{2M} [p - eA_1(x)]^2 + V(x) \tag{1.24}$$

where M is the mass of the particle, e is its charge, and we shall require henceforth that

$$A_1(x) \simeq A_1(0) \tag{1.25}$$

[We shall, in fact, replace $A_1(x)$ in (1.24) by $A_1(0)$.] For polynomially bounded potentials $V \geqslant 0$ and arbitrary $A_1(0) \in \mathbb{R}$, H_p determines a self-adjoint operator defined on a domain dense in $\mathscr{H}_p := L^2(\mathbb{R}, dx)$.

The dynamics of the field A_1, for $e = 0$, is given by a free-field Hamiltonian H_f defined on some domain dense in the Fock space $\mathscr{F}$ of states of the field oscillators by

$$H_f = \int dk\, |k|\, a^*(k)\, a(k) \tag{1.26}$$

where a^* and a are standard creation and annihilation operators. The operator H_f is positive and self-adjoint, and its ground state is the usual Fock vacuum φ_0. Expressed in terms of creation and annihilation operators, A_1 is given by

$$A_1(x, t) = \frac{1}{2\pi} \int dk \frac{\vartheta(k)}{(2\,|k|)^{1/2}} \{a^*(k)\, e^{i(|k|\,t - k\cdot x)} + \text{h.c.}\} \tag{1.27}$$

where ϑ (the ultraviolet cutoff) is a real test function with $\vartheta(0) = 1$. If the support of ϑ is concentrated around $k = 0$, then (1.25) holds. One easily checks that (1.26) and (1.27) yield the formula

$$\begin{aligned} &\langle A_1(0)\, \varphi_0, (e^{-|t|\,H_f} - 1)\, A_1(x)\, \varphi_0\rangle \\ &\quad = \frac{1}{4\pi^2} \int d\omega\, (e^{i\omega t} - 1) \int dk\, \vartheta(k)^2 \frac{e^{-ik\cdot x}}{\omega^2 + k^2} \end{aligned} \tag{1.28}$$

In particular,

$$\langle A_1(0)\, \varphi_0, (e^{-|t|\,H_f} - 1)\, A_1(0)\, \varphi_0\rangle = \int d\rho(\omega)(e^{i\omega t} - 1) \tag{1.29}$$

where

$$d\rho(\omega) \sim \frac{d\omega}{\omega} \qquad \text{for} \quad \omega \approx 0 \qquad \left(\frac{1}{f}\,\text{distribution}\right) \tag{1.30}$$

We are interested in the dynamics of the coupled system with Hamiltonian H given by

$$H := H_p + H_f \tag{1.31}$$

acting on the Hilbert space

$$\mathscr{H} := \mathscr{H}_p \otimes \mathscr{F} \tag{1.32}$$

One of the simplest questions one can ask about this system is whether H has a normalizable ground state in $\mathscr{H}$. Let us assume that V has a minimum at $x = 0$, and let $\delta_0(x)$ denote the δ-function at $x = 0$. If one exists, a ground state Ω of H can be obtained as the limit

$$\Omega = \lim_{t \to +\infty} \Phi(t) \tag{1.33}$$

where

$$\Phi(t) = \frac{e^{-tH}(\delta_0 \otimes \varphi_0)}{\|e^{-tH}(\delta_0 \otimes \varphi_0)\|} \tag{1.34}$$

where $\|(\cdot)\|$ denotes the norm on $\mathscr{H}$. Note that, although the norm of $\delta_0 \otimes \varphi_0$ is infinite, the norms of the vectors $e^{-tH}(\delta_0 \otimes \varphi_0)$ are finite, for $t > 0$.

The states $\Phi(t)$ can be studied with the help of the *Feynman–Kac formula.* In particular, for an arbitrary function F of x, we deduce from (1.24), (1.26), and (1.31) that

$$\langle \Phi(t), F\Phi(t) \rangle$$

$$= Z_{[-t,t]}^{-1} \int dW_{[-t,t]}(x(\cdot))$$

$$\times \exp\left[-\int_{-t}^{t} V(x(\tau))\, d\tau \right] E\left(\exp\left[ie \int_{-t}^{t} A_1(\tau)\, \dot{x}(\tau)\, d\tau \right] \right) F(x(0)) \tag{1.35}$$

where $dW_{[-t,t]}$ is the Wiener measure on the space of Brownian paths $x(\tau)$, $\tau \in [-t, t]$, with $x(-t) = x(t) = 0$, the functional E is the imaginary-time (Euclidean) vacuum expectation on configurations of the field A_1, and $Z_{[-t,t]}$ is a "partition function." Using the fact that E is Gaussian, we conclude that E is completely determined by the "covariance matrix" (1.28), and hence

$$E\left(\exp\left[ie \int_{-t}^{t} A_1(\tau)\, \dot{x}(\tau)\, d\tau \right] \right)$$

$$= \exp\left\{ -\frac{e^2}{2} \int_{-t}^{t} \int [x(\tau) - x(\sigma)]^2\, g(\tau - \sigma)\, d\tau\, d\sigma \right\} \tag{1.36}$$

where

$$g(\tau) = \int \omega^2 e^{i\omega\tau}\, d\rho(\omega)$$

$$\sim \text{const} \cdot \frac{1}{1 + \tau^2} \qquad \text{as} \quad |\tau| \to \infty \tag{1.37}$$

Thus, for

$$V(x) = \lambda \cos(2\pi x) \tag{1.38}$$

the expectation (1.35) approaches that of the continuum limit of the discrete Gaussian chain, with $1/r^2$ interaction energy, as $\lambda \to \infty$. Also, for $V(x) = \lambda(x^2 - 1)^2$, the expectation (1.35) approaches that of the continuum limit of the Ising chain with $1/r^2$ interaction energy, as $\lambda \to \infty$.

In order to simplify our problems, we shall discretize the imaginary-time variable τ: $\tau \in \mathbb{R}$ is replaced by $\tau \in \mathbb{Z}$, and $x(\tau = j)$ is denoted by n_j, for $j \in \mathbb{Z}$. Although our analytical methods *are applicable* to the models with λ

finite, but large, we shall only study the limiting models, with $\lambda \to \infty$. After these simplifications, we shall prove, in Sections 3 and 4, that the model with V given by (1.36), $\lambda \to \infty$, exhibits a depinning (roughening) transition as $\beta = e^2$ is lowered. Physically, this means that the Hamiltonian H given in (1.31), with V as in (1.38) (λ large enough), has a normalizable ground state Ω localized near $x = 0$, provided e^2 is large enough, while the (generalized) ground state of H is *extended* for small values of e^2. [For $V(x) = \lambda(x^2 - 1)^2$, the results in refs. 1 and2 suggest that the ground state of H is doubly degenerate, and the symmetry $x \to -x$ spontaneously broken, for large values of e^2, while it is unique for small e^2.]

It would be more interesting to study properties of charge transport, as described by e^{-itH}, for different values of e^2, but our methods are inadequate for that task.

Remark. The quantum mechanical system described by (1.24)–(1.32) is essentially equivalent to one with Hamiltonian

$$H = H_p + H_f - \text{const}$$

where

$$H_p = -\frac{\Delta}{2M} + V(x) + e\varphi(x)$$

with $\varphi(x)$ given by the rhs of (1.27) (at $t = 0$), and H_f as in (1.26), in the approximation where $\varphi(x)$ is replaced by

$$\varphi(0) + \left(\frac{d}{dx}\varphi\right)(0) \cdot x$$

One might want to interpret φ as an electric potential acting on the particle. In view of (1.29) and (1.30) one could say that φ describes quantum mechanical $1/f$ noise.

For some background material on the problems described in this section see ref. 8.

2. PRELIMINARY CONSIDERATIONS ON THE LOW-TEMPERATURE PHASE

2.1. Description of the Problem

For $L \in \mathbb{N}$, let $\Lambda \equiv [-L, L] \cap \mathbb{Z}$ and $\Lambda^c \equiv \mathbb{Z} \backslash \Lambda$. Let $\Omega \equiv (\mathbb{Z})^{\mathbb{Z}}$ and let Ω_Λ be a subset of configurations $n \equiv (n_i)_{i \in \mathbb{Z}}$ defined by

$$\Omega_\Lambda := \{n \in \Omega : n_i \equiv 0 \text{ for } i \in \Lambda^c\} \tag{2.1}$$

We consider a system specified by a Hamiltonian function defined on Ω_Λ as follows:

$$H(n) := \frac{1}{2} \sum_{i,j \in \mathbb{Z}} g(i-j)(n_i - n_j)^2 \tag{2.2}$$

for a (positive) $g(i-j) \sim |i-j|^{-2}$ for $|i-j|$ large. Under the above assumptions we say that the system has zero (Ising) boundary conditions, as compared to adiabatic (Dirichlet) boundary conditions defined by the additional restriction

$$g(i-j) \equiv 0 \qquad \text{if} \quad |i| \quad \text{or} \quad |j| > L+1 \tag{2.3}$$

For $\beta > 0$, a finite-volume Gibbs measure on Ω_Λ is given by

$$\mu_{\Lambda,\beta}(F) = \frac{\sum_{n \in \Omega_\Lambda} [e^{-\beta H(n)} F(n)]}{\sum_{n \in \Omega_\Lambda} (e^{-\beta H(n)})} \tag{2.4}$$

By simple arguments, the measure $\mu_{\Lambda,\beta}$ defines a unique probability measure on Ω, denoted by the same symbol.

Our problem is to show that, for $\beta > \beta_0$, for some sufficiently large $0 < \beta_0 < \infty$, the sequence $\{\mu_{\Lambda,\beta}\}$ converges, as $\Lambda \nearrow \mathbb{Z}$, and that the expectation (2.4), for F an exponential function, is uniformly bounded in Λ. The solution of this problem is based on energy–entropy arguments similar to ones in ref. 1 for the $1/r^2$ Ising model.

2.2. Configurations and Contours

Let $\Omega_{\text{Ising}}(\Lambda)$ consist of sequences $\sigma \equiv (\sigma_i)_{i \in \mathbb{Z}}$ of spins such that $\sigma_i \in \{-1, +1\}$, and if $i \in \Lambda^c$, then $\sigma_i \equiv -1$. Let $\mathbb{Z}^*$ be the set of bonds of the lattice $\mathbb{Z}$. Identifying $b \in \mathbb{Z}^*$ with its midpoint, we have $\mathbb{Z}^* \approx \mathbb{Z} + 1/2$. We set $\mathbb{Z}^*_\Lambda = \mathbb{Z}^* \cap (\Lambda \cup \partial \Lambda^c)^*$. Each configuration $\sigma \in \Omega_{\text{Ising}}(\Lambda)$ defines a unique even subset $\Gamma \equiv \Gamma(\sigma) \subset \mathbb{Z}^*_\Lambda$ as follows:

$$b \in \Gamma \qquad \text{iff} \quad \sigma_i \sigma_{i+1} = -1 \tag{2.5}$$

and $\#\Gamma(\sigma)$ is even. Conversely, each even subset $\Gamma \subseteq \mathbb{Z}^*_\Lambda$ of spin flips determines a unique configuration $\sigma \equiv \sigma(\Gamma) \in \Omega_{\text{Ising}}(\Lambda)$.

For a subset $\gamma \subset \Gamma$ let $b_-(\gamma)$ be the smallest and $b_+(\gamma)$ the largest bond belonging to γ. The diameter $d(\gamma)$ of γ is, by definition, equal to $b_+(\gamma) - b_-(\gamma) + 1$. Let $I(\gamma) \equiv \mathbb{Z}^* \cap [b_-(\gamma), b_+(\gamma)]$ and $I_\pm(\gamma) \equiv \{i \in I(\gamma)^*: \sigma_i(\gamma) = \pm 1\}$.

Each set of spin flips $\Gamma \subseteq \mathbb{Z}_\Lambda^*$ ($\#\Gamma$ even) can be partitioned into primitive contours $\{\gamma_{\alpha_1},..., \gamma_{\alpha_r}\}$ satisfying the distance condition (D) of ref. 1:

(D a) $\#\gamma_\alpha$ is even,

$$\gamma_\alpha \cap \gamma_{\alpha'} = \varnothing \qquad \text{if} \quad \alpha \neq \alpha', \quad \bigcup_\alpha \gamma_\alpha = \Gamma$$

(D b) $\text{dist}(\gamma_\alpha, \gamma_{\alpha'}) \geqq M[\min(d(\gamma_\alpha), d(\gamma_{\alpha'}))]^{3/2}$ for $\alpha \neq \alpha'$ (2.6)

(D c) If γ is a subset of spin flips in γ_α satisfying

$$\text{dist}(\gamma, \gamma_\alpha \backslash \gamma) \geqq 2Md(\gamma)^{3/2}, \qquad \text{then } \#\gamma \text{ is odd} \tag{2.7}$$

The constant $M > 1$ is assumed to be sufficiently large, but independent of Γ and Λ. Following ref. 2, we can and do assume that the partition $\{\gamma_\alpha\}$ of Γ given above is (unique and) maximal. (Then the contours γ_α are called irreducible.)

For $N \in \mathbb{N}$, we define the following characteristic functions on $\mathbb{Z}$:

$$\chi_{+1}^{(N)}(n_i) \equiv \begin{cases} 1 & \text{for} \quad |n_i| \geqq 2N \\ 0 & \text{otherwise} \end{cases} \tag{2.8}$$

and

$$\chi_{-1}^{(N)}(n_i) \equiv 1 - \chi_{+1}^{(N)}(n_i) \tag{2.9}$$

To each Γ we associate a unique subset of Ω_Λ, i.e., of n-configurations, on a level $N \in \mathbb{N}$, defined by the characteristic function

$$\chi_\Gamma^{(N)}(n) \equiv \prod_{i \in \Lambda} \chi_{\sigma_i(\Gamma)}^{(N)}(n_i) \tag{2.10}$$

We say that the sets Γ_α and $\Gamma_{\alpha'}$ are compatible, writing $\Gamma_\alpha \succcurlyeq \Gamma_{\alpha'}$, iff

$$\sigma_i(\Gamma_\alpha) \geqq \sigma_i(\Gamma_{\alpha'}), \qquad i \in \Lambda \tag{2.11}$$

We note that if $\Gamma_{\alpha_1} \succcurlyeq \Gamma_{\alpha_2} \succcurlyeq \cdots \succcurlyeq \Gamma_{\alpha_N}$, then

$$\chi_{\Gamma_{\alpha_1},..., \Gamma_{\alpha_N}}(n) \equiv \prod_{K=1}^{N} \chi_{\alpha_K}^{(K)}(n) \neq 0 \tag{2.12}$$

where $\chi_{\alpha_K}^{(K)}(n)$ is given by (2.10) for $\Gamma = \Gamma_{\alpha_K}$ and we say that the family $\{\Gamma_{\alpha_1},..., \Gamma_{\alpha_N}\}$ is compatible.

2.3. A Short Review of the $1/r^2$ Ising Model

Theorem A of ref. 1 says that, for any primitive contour γ in a maximal partition of a configuration Γ of spin flips satisfying condition (D), for some sufficiently large $M > 1$, the following lower bound is fulfilled:

$$H_{\text{Ising}}(\Gamma) - H_{\text{Ising}}(\Gamma \backslash \gamma) \geqq 2\bar{c} H_{\text{Ising}}(\gamma) \tag{2.13}$$

with a constant $\bar{c} > 0$ independent of Γ, γ, and Λ.

Let $\{i_K \in \mathbb{Z} + 1/2: K = 1,..., r\}$ be an increasing finite sequence specifying a set γ of spin flips. The logarithmic length $L(\gamma)$ of γ is defined by

$$L(\gamma) := \sum_{K=1,...,r} \{[\ln_2(i_{K+1} - i_K)] + 1\} \tag{2.14}$$

From ref. 1 one has the following estimations:

(a) *An energy estimate*: If M is sufficiently large, then there is a constant $C_1 > 0$, independent of the choice of primitive contours γ, such that

$$H_{\text{Ising}}(\gamma) \geqq C_1 L(\gamma) \tag{2.15}$$

(b) *Entropy estimate*: There is a number $C_2 > 0$, independent of Λ and the choice of a point $i \in \Lambda$, such that, for any $R \in \mathbb{N}$,

$$\#\{\gamma = \text{primitive contour: } i \in I(\gamma),\ L(\gamma) < R\} \leqq e^{C_2 R} \tag{2.16}$$

The standard way, for bounded spin systems, to get an estimate on the probability of a primitive contour γ to be present involves using the *Peierls transform*, defined as the map

$$\begin{aligned} *\colon\ \{\Gamma(\sigma): \sigma \in \Omega_{\text{Ising}}(\Lambda), \gamma \subset \Gamma(\sigma)\} &\to \{\Gamma(\sigma): \sigma \in \Omega_{\text{Ising}}(\Lambda)\} \\ \Gamma \mapsto \Gamma^* &:= \Gamma \backslash \gamma \end{aligned} \tag{2.17}$$

Using the definition (2.17) we have (with $H \equiv H_{\text{Ising}}$),

$$\begin{aligned} P_{\Lambda,\beta}(\gamma) &\equiv \frac{\sum_{\sigma: \gamma \subset \Gamma(\sigma)} e^{-\beta H(\sigma)}}{\sum_\sigma e^{-\beta H(\sigma)}} = \frac{\sum_{\Gamma: \gamma \subset \Gamma} e^{-\beta H(\Gamma)}}{\sum_\Gamma e^{-\beta H(\Gamma)}} \\ &\leqq \frac{\sum_{\Gamma: \gamma \subset \Gamma} e^{-\beta H(\Gamma)}}{\sum_{\Gamma^*: \gamma \subset \Gamma} e^{-\beta H(\Gamma^*)}} \end{aligned} \tag{2.18}$$

Writing the rhs of (2.18) in the form

$$\text{rhs}(2.18) = \frac{\sum_{\Gamma: \gamma \subset \Gamma} e^{-\beta H(\Gamma^*)} e^{-\beta(H(\Gamma) - H(\Gamma^*))}}{\sum_{\Gamma: \gamma \subset \Gamma} e^{-\beta H(\Gamma^*)}} \tag{2.19}$$

and using (2.13), we get the following *Peierls estimate*:

$$P_{\Lambda,\beta}(\gamma) \leqq e^{-\bar{c}\beta H_{\mathrm{Ising}}(\gamma)} \tag{2.20}$$

Then, using (2.15) together with the entropy estimate (2.16), we conclude that, for $\beta > \beta_0$ with $\beta_0 > 0$ large enough,

$$P_{\Lambda,\beta}(\sigma_i = 1) \equiv \sum_{\gamma:\, i \in I(\gamma)} P_{\Lambda,\beta}(\gamma) \leqq e^{-\tilde{c}\beta} \tag{2.21}$$

for some constant $\tilde{c} > 0$ independent of Λ, $i \in \Lambda$, and β.

3. PEIERLS ESTIMATE FOR THE DG MODEL AT LOW TEMPERATURES

3.1. Basic Ideas

Consider first the problem of estimating the probability of the event $\{n \in \Omega_\Lambda\colon |n_i| \geqq 2\}$ for some arbitrary $i \in \Lambda$ defined with the measure $\mu_{\Lambda,\beta}$. Using the notation of Section 2.2, we have that

$$\begin{aligned}\mu_{\Lambda,\beta}\{n \in \Omega_\Lambda\colon |n_i| \geqq 2\} &\equiv \mu_{\Lambda,\beta}\chi^{(1)}_{+1}(n_i)\\ &= \sum_{\Gamma:\, \sigma_i(\Gamma) = +1} \mu_{\Lambda,\beta}\chi^{(1)}_{\Gamma}(n)\end{aligned} \tag{3.1}$$

The last sum can be represented using primitive contours $\{\gamma, \sigma_i(\gamma) = +1\}$ as follows:

$$\sum_{\gamma:\, \sigma_i(\gamma) = +1} \sum_{\Gamma:\, \gamma \subset \Gamma} \mu_{\Lambda,\beta}\chi^{(1)}_{\Gamma}(n) \equiv \sum_{\gamma:\, \sigma_i(\gamma) = +1} P^{(1)}_{\Lambda,\beta}(\gamma) \tag{3.2}$$

Our task is to establish a Peierls estimate, i.e., to prove an upper bound on the probability $P^{(1)}_{\Lambda,\beta}(\gamma)$ of existence of a primitive contour γ. Since our configuration space Ω_Λ is noncompact, we use *a modified Peierls transform* in order to reach that goal: For $\{\Gamma\colon \gamma \subset \Gamma\}$, we define a map

$$\chi^{(1)}_{\Gamma}(n) \mapsto \tilde{\chi}^{(1)}_{\Gamma}(n) := \prod_{i \in \Lambda \setminus I(\gamma)} \chi^{(1)}_{\sigma(i)}(n_i) \prod_{i \in I_-(\gamma)} \chi^{(1)}_{-1}(n_i) \prod_{i \in I_+(\gamma)} \tilde{\chi}^{(1)}_{+1}(n_i) \tag{3.3}$$

where

$$\tilde{\chi}^{(1)}_{+1}(n_i) \equiv \begin{cases} 1 & \text{for } |n_i| \geqq 1 \\ 0 & \text{otherwise} \end{cases} \tag{3.4}$$

By virtue of the simple fact that

$$0 < \sum_{\Gamma:\, \Gamma \supset \gamma \text{ fixed}} \tilde{\chi}_\Gamma^{(1)} \leqq 1 \tag{3.5}$$

we get that

$$P_{\Lambda,\beta}^{(1)}(\gamma) \leqslant \frac{\sum_{\Gamma:\,\gamma\subset\Gamma} \mu_{\Lambda,\beta}\chi_\Gamma}{\sum_{\Gamma:\,\gamma\subset\Gamma} \mu_{\Lambda,\beta}\tilde{\chi}_\Gamma} = \frac{\sum_{\Gamma:\,\gamma\subset\Gamma} \mu_{\Lambda,\beta}\chi_\Gamma}{\sum_{\Gamma:\,\gamma\subset\Gamma} \mu_{\Lambda,\beta}\chi_\Gamma(\mu_{\Lambda,\beta}\tilde{\chi}_\Gamma/\mu_{\Lambda,\beta}\chi_\Gamma)} \tag{3.6}$$

We see that in order to get our Peierls estimate, we have to prove a uniform (in Γ) bound from below on the factors $(\mu_{\Lambda,\beta}\tilde{\chi}_\Gamma/\mu_{\Lambda,\beta}\chi_\Gamma)$. For that purpose, for a given Γ, $\gamma \subset \Gamma$, we change variables in $\mu_{\Lambda,\beta}\tilde{\chi}_\Gamma$, by passing from the variables $\{\tilde{n} \in \operatorname{supp} \tilde{\chi}_\Gamma\}$ to the variables $\{n \in \operatorname{supp} \chi_\Gamma\}$, defined by

$$\begin{aligned} \tilde{n}_i &\equiv n_i \qquad && \text{for} \quad i \in \Lambda \backslash I_+(\gamma) \\ \tilde{n}_i &\equiv n_i - s_i \qquad && \text{for} \quad i \in I_+(\gamma) \end{aligned} \tag{3.7}$$

where $s_i \equiv \operatorname{sign} n_i$.

Then we have that

$$\begin{aligned} H(\tilde{n}_i) = H(n_i) &- \sum_{\substack{i\in I_+(\gamma)\\ j\in\Lambda\backslash I_+(\gamma)}} g(i-j)[2(n_i-n_j)\, s_i - 1] \\ &- \frac{1}{2} \sum_{i,j\in I_+(\gamma)} g(i-j)[2(n_i-n_j)(s_i-s_j) - (s_i-s_j)^2] \end{aligned} \tag{3.8}$$

Using this identity and defining μ_Γ by

$$\mu_\Gamma(F) := \frac{\mu_{\Lambda,\beta}(\chi_\Gamma \cdot F)}{\mu_{\Lambda,\beta}(\chi_\Gamma)} \tag{3.9}$$

we obtain from Jensen's inequality the following lower bound:

$$\begin{aligned} \frac{\mu_{\Lambda,\beta}\tilde{\chi}_\Gamma}{\mu_{\Lambda,\beta}\chi_\Gamma} \geqq \exp\Bigg\{\beta &\sum_{\substack{i\in I_+(\gamma)\\ j\in\Lambda\backslash I_+(\gamma)}} g(i-j)\, \mu_\Gamma(2(n_i-n_j)\, s_i - 1) \\ &+ \beta\frac{1}{2} \sum_{i,j\in I_+(\gamma)} g(i-j)\, \mu_\Gamma(2(n_i-n_j)(s_i-s_j) - (s_i-s_j)^2)\Bigg\} \end{aligned} \tag{3.10}$$

Let us analyze the exponent on the rhs of (3.10): First we note that if i, $j \in I_+(\gamma)$, i.e., $|n_i|$, $|n_j| \geqq 2$, then

$$2(n_i-n_j)(s_i-s_j) - (s_i-s_j)^2 \geqq 0 \tag{3.11}$$

Next, for $i \in I_+(\gamma)$ and $j \in I_-(\Gamma)$ $[\sigma_j(\Gamma) = -1]$, i.e., $|n_i| \geqq 2$, $|n_j| \leqq 1$, we have that

$$2(n_i - n_j)\, s_i - 1 = 2(|n_i| - s_i n_j) - 1 \geqq 1 \tag{3.12}$$

Hence

$$\sum_{\substack{i \in I_+(\gamma) \\ j \in \Lambda \backslash I_+(\gamma), \sigma_j(\Gamma) = -1}} g(i-j)\, \mu_\Gamma (2(n_i - n_j)\, s_i - 1) \geqq H_{\text{Ising}}(\Gamma) - H_{\text{Ising}}(\Gamma \backslash \gamma) \tag{3.13}$$

and, by virtue of (2.13),

$$\sum_{\substack{i \in I_+(\gamma) \\ j \in \Lambda \backslash I_+(\gamma), \sigma_j(\Gamma) = -1}} g(i-j)\, \mu_\Gamma (2(n_i - n_j)\, s_i - 1) \geqq \bar{c} H_{\text{Ising}}(\gamma) \tag{3.14}$$

for some constant $\bar{c} > 0$ independent of γ (Γ, Λ, and β). The last part of the first sum in the exponent of the rhs of (3.10) is bounded from below as follows:

$$\begin{aligned} &\sum_{\substack{i \in I_+(\gamma) \\ j \in I_+(\Gamma \backslash \gamma)}} g(i-j)\, \mu_\Gamma (2(n_i - n_j)\, s_i - 1) \\ &\qquad \geqq - \sum_{\substack{i \in I_+(\gamma) \\ j \in I_+(\Gamma \backslash \gamma)}} g(i-j)\, 2[\mu_\Gamma (|n_j| - 2) + 1] \end{aligned} \tag{3.15}$$

Here we use the fact that only configurations for which

$$(n_i - n_j)\, s_i = |n_i| - s_i n_j \leqq 0 \tag{3.16}$$

are dangerous for us. But (3.16) can happen only if $s_i = s_j$. However, in this case

$$|n_i| - s_i n_j = (|n_i| - 2) - (|n_j| - 2) \geqq -(|n_j| - 2) \tag{3.17}$$

[since $i \in I_+(\gamma)$, so that $|n_i| - 2 \geqq 0$]. Note that, under our conditions, one can expect that $\mu_\Gamma(|n_j| - 2)$ is smaller than $\mu_\Gamma |n_j|$. Combining (3.10)–(3.15), we conclude the following Lemma.

Lemma 3.1. For any Γ containing some fixed contour γ,

$$\frac{\mu_{\Lambda,\beta} \tilde{\chi}_\Gamma}{\mu_{\Lambda,\beta} \chi_\Gamma} \geqq \exp\left\{\beta \left(c H_{\text{Ising}}(\gamma) - \sum_{\substack{i \in I_+(\gamma) \\ j \in I_+(\Gamma \backslash \gamma)}} g(i-j)\, \mu_\Gamma (|n_j| - 2) \right)\right\} \tag{3.18}$$

for some constant $c \in (0, \bar{c})$ independent of γ (Γ, Λ) and β [provided $M > 1$ in the definition of condition (D) is chosen sufficiently large]. ▌

Our Peierls estimate would be complete if we were able to prove a bound

$$\mu_\Gamma(|n_j| - 2) < C \tag{3.19}$$

for some constant $C > 0$ independent of $j \in I_+(\Gamma \backslash \gamma)$, γ, Γ, Λ, the constant M, and β.

Let us remark that then, by choosing the constant M sufficiently large, we may have an estimate from below of the exponent on the rhs of (3.18) by $\beta(c/2) H_{\mathrm{Ising}}(\gamma)$.

Let us now start our proof of (3.19). First, we note the simple fact that, for $j \in \Lambda$, $\sigma_j(\Gamma) = +1$,

$$\begin{aligned} \mu_\Gamma(|n_j| - 2) &= \{\mu_\Gamma[\chi(|n_j| \leqq 3) \cdot (|n_j| - 2)] + \mu_\Gamma[\chi(|n_j| \geqq 4)] 2\} \\ &\quad + \mu_\Gamma(\chi(|n_j| \geqq 4)(|n_j| - 4)) \\ &\leqslant 2 + \mu_\Gamma(\chi(|n_j| \geqq 4)(|n_j| - 4)) \end{aligned} \tag{3.20}$$

The second term on the rhs of (3.20) can be represented with the help of contours on level $N = 2$ compatible with Γ. We then have

$$\begin{aligned} &\mu_\Gamma(\chi(|n_j| \geqq 4)(|n_j| - 4)) \\ &\quad = \sum_{\gamma' : \sigma_j(\gamma') = +1} \left[\sum_{\Gamma' : \Gamma \succcurlyeq \Gamma', \gamma' \subset \Gamma'} \mu_\Gamma(\chi_{\Gamma'}^{(2)} \cdot (|n_j| - 4)) \right] \end{aligned} \tag{3.21}$$

with $\{\gamma'\}$ ranging over primitive contours and the convention that if $\{\Gamma' : \Gamma \succcurlyeq \Gamma', \gamma' \subset \Gamma'\} = \varnothing$, then the corresponding sum in square brackets is zero.

Writing

$$\sum_{\Gamma' : \Gamma \succcurlyeq \Gamma', \Gamma' \supset \gamma'} \mu_\Gamma(\chi_{\Gamma'}^{(2)} \cdot (|n_j| - 4)) \equiv \sum_{\Gamma' : \Gamma \succcurlyeq \Gamma', \gamma' \subset \Gamma'} \mu_\Gamma(\chi_{\Gamma'}^{(2)}) \, \mu_{\Gamma, \Gamma'}(|n_j| - 4) \tag{3.22}$$

and inserting this identity into (3.21), we derive from (3.20) the inequality

$$\mu_\Gamma(|n_j| - 2) \leqq 2 + \sum_{\gamma'} \left(\sum_{\Gamma' : \Gamma \succcurlyeq \Gamma', \gamma' \subset \Gamma'} \mu_\Gamma(\chi_{\Gamma'}^{(2)}) \, \mu_{\Gamma, \Gamma'}(|n_j| - 4) \right) \tag{3.23}$$

Using this inequality, we observe that if we had a bound

$$\mu_{\Gamma, \Gamma''}(|n_{j''}| - 4) < C' \tag{3.24}$$

for some constant $C'>0$ independent of $\Gamma''(\gamma', \Lambda, \beta, M)$ and j'', $\sigma_{j''}(\Gamma'') = +1$, then, by the arguments analogous to those in (3.17)–(3.18), we would be able to complete our Peierls estimate for the primitive contours γ'. Having such an estimate for

$$P_\Gamma(\gamma') \equiv \sum_{\Gamma':\, \Gamma \succcurlyeq \Gamma',\, \gamma' \subset \Gamma'} \mu_\Gamma(\chi_{\Gamma'}^{(2)})$$

(which is of the same form as for the corresponding Ising model) and using (3.23)–(3.24) then, for sufficiently small temperatures β^{-1}, we would obtain the upper bound

$$\mu_\Gamma(|n_j| - 2) \leqq 2 + \mu_\Gamma(\chi(|n_j| \geqq 4)) \cdot C' \tag{3.25}$$

Since the rhs of (3.25) is independent of Λ, β, and, moreover, of Γ and j provided the temperature is sufficiently small, we are then able to complete our Peierls estimate on the level below. We propose to iterate this procedure. We will show that, at some sufficiently high level (depending on Λ), a bound similar to (3.24) is easy to prove. From that we will simultaneously get a Peierls estimate at every level and a bound uniform in Λ as well as in $j \in \Lambda$ for $\mu_{\Lambda,\beta} |n_j|$. This will be carried out in the next section. After generalizing our arguments in the proof of the bound on $\mu_{\Lambda,\beta} |n_j|$, we will get in Section 3.3 the exponential bound and restricted analyticity properties.

3.2. The Peierls Estimate Completed

We start this section by proving two technical lemmas that will be needed to prove the Peierls estimate on an arbitrary level.

For a set Γ of spin flips in $\Omega_{\text{Ising}}(\Lambda)$, let

$$\mu_\Gamma(F) := \frac{\mu_{\Lambda,\beta}(\chi_\Gamma^{(1)} \cdot F)}{\mu_{\Lambda,\beta}(\chi_\Gamma^{(1)})} \tag{3.26}$$

and for any compatible family $\{\Gamma_{\alpha(1)}, ..., \Gamma_{\alpha(N)}\}$, $N \in \mathbb{N}$, $N \geqq 2$, we define recursively

$$\mu_{\Gamma_{\alpha(1)}, ..., \Gamma_{\alpha(N-1)}, \Gamma_{\alpha(N)}}(F) := \frac{\mu_{\Gamma_{\alpha(1)}, ..., \Gamma_{\alpha(N-1)}}(\chi_{\Gamma_{\alpha(N)}}^{(N)} \cdot F)}{\mu_{\Gamma_{\alpha(1)}, ..., \Gamma_{\alpha(N-1)}}(\chi_{\Gamma_{\alpha(N)}}^{(N)})} \tag{3.27}$$

If γ is a primitive contour contained in Γ, then we define

$$\tilde{\chi}_\Gamma^{(N)}(n) := \prod_{i \in \Lambda \setminus I_+(\gamma)} \chi_{\sigma_i(\Gamma)}^{(N)}(n_i) \cdot \prod_{i \in I_+(\gamma)} \tilde{\chi}_+^{(N)}(n_i) \tag{3.28}$$

with

$$\tilde{\chi}_{+}^{(N)}(n_i) = \begin{cases} 1 & \text{for} \quad |n_i| \geqq 2N-1 \\ 0 & \text{otherwise} \end{cases} \tag{3.29}$$

Lemma 3.1. For any compatible family $\{\Gamma_{\alpha(1)},..., \Gamma_{\alpha(N)}\}$ and any primitive contour $\gamma \subseteq \Gamma_{\alpha(N)}$,

$$\frac{\mu_{\Gamma_{\alpha(1)},..., \Gamma_{\alpha(N-1)}}(\tilde{\chi}^{(N)}_{\Gamma_{\alpha(N)}})}{\mu_{\Gamma_{\alpha(1)},..., \Gamma_{\alpha(N-1)}}(\chi^{(N)}_{\Gamma_{\alpha(N)}})}$$

$$\geqq \exp\left\{\beta\left(cH_{\text{Ising}}(\gamma) - 2 \sum_{\substack{i \in I_+(\gamma) \\ j \in I_+(\Gamma_{\alpha(N)} \setminus \gamma)}} g(i-j)\, \mu_{\Gamma_{\alpha(1)},..., \Gamma_{\alpha(N)}}(|n_j| - 2N)\right)\right\} \tag{3.30}$$

for some positive $c < \bar{c}$ independent of N, γ, $\{\Gamma_{\alpha(1)},..., \Gamma_{\alpha(N)}\}$, Λ, and β. ∎

Remark. The proof is similar to the one of Lemma 3.1.

Proof. Let us define

$$\tilde{\chi}_{\Gamma_{\alpha(1)},..., \Gamma_{\alpha(N)}} \equiv \left(\prod_{K=1}^{N-1} \chi_{\Gamma_{\alpha(K)}}\right) \tilde{\chi}_{\Gamma_{\alpha(N)}}$$

Instead of the summation variables $\{\tilde{n} \in \operatorname{supp} \tilde{\chi}_{\Gamma_{\alpha(1)},..., \Gamma_{\alpha(N)}}\}$, we define new summation variables $\{n \in \operatorname{supp} \chi_{\Gamma_{\alpha(1)},..., \Gamma_{\alpha(N)}}\}$ in the numerator of the lhs of (3.30) by requiring that

$$\begin{aligned} \tilde{n}_i &\equiv n_i && \text{if} \quad i \in \Lambda \setminus I_+(\gamma) \\ \tilde{n}_i &\equiv n_i - s_i && \text{if} \quad i \in I_+(\gamma) \end{aligned} \tag{3.31}$$

with $s_i \equiv \operatorname{sign}(n_i)$.

Then

$$H(\tilde{n}) = H(n) - \sum_{\substack{i \in I_+(\gamma) \\ j \in \Lambda \setminus I_+(\gamma)}} g(i-j)[2(n_i - n_j)\, s_i - 1]$$

$$- \frac{1}{2} \sum_{i,j \in I_+(\gamma)} g(i-j)[2(n_i - n_j)(s_i - s_j) - (s_i - s_j)^2] \tag{3.32}$$

Inserting this equation in the lhs of (3.30) and using Jensen's inequality, we get the following lower bound:

$$\frac{\mu_{\Gamma_{\alpha(1)},\ldots,\Gamma_{\alpha(N-1)}}\tilde{\chi}^{(N)}_{\Gamma_{\alpha(N)}}}{\mu_{\Gamma_{\alpha(1)},\ldots,\Gamma_{\alpha(N-1)}}\chi^{(N)}_{\Gamma_{\alpha(N)}}}$$

$$\geqq \exp\left\{\frac{\beta}{2}\left[2\sum_{\substack{i\in I_+(\gamma)\\ j\in \Lambda\setminus I_+(\gamma)}} g(i-j)\,\mu_{\Gamma_{\alpha(1)},\ldots,\Gamma_{\alpha(N)}}(2(n_i-n_j)\,s_i-1)\right.\right.$$

$$\left.\left.+\sum_{i,j\in I_+(\gamma)} g(i-j)\,\mu_{\Gamma_{\alpha(1)},\ldots,\Gamma_{\alpha(N)}}(2(n_i-n_j)(s_i-s_j)-(s_i-s_j)^2)\right]\right\} \tag{3.33}$$

Since, for $i, j\in I_+(\gamma)$, $\gamma\subset\Gamma_{\alpha(N)}$, we have that $|n_i|, |n_j|\geqq 2N$, it follows that

$$2(n_i-n_j)(s_i-s_j)-(s_i-s_j)^2\geqq 0, \qquad i, j\in I_+(\gamma) \tag{3.34}$$

Moreover, if $\sigma_j(\Gamma_{\alpha(N)})=-1$, then $|n_j|\leqq 2N-1$, and so, for $|n_i|\geqq 2N$, i.e., $\sigma_i(\Gamma_{\alpha(N)})=+1$, we have that

$$2(n_i-n_j)\,s_i-1=2(|n_i|-s_in_j)-1\geqq 1 \tag{3.35}$$

Hence

$$\sum_{\substack{i\in I_+(\gamma)\\ j\in \Lambda\setminus I_+(\gamma):\,\sigma_j(\Gamma_{\alpha(N)})=-1}} g(i-j)\,\mu_{\Gamma_{\alpha(1)},\ldots,\Gamma_{\alpha(N)}}(2(n_i-n_j)\,s_i-1)$$

$$\geqq H_{\text{Ising}}(\Gamma_{\alpha(N)})-H_{\text{Ising}}(\Gamma_{\alpha(N)}\setminus\gamma)$$

$$\geqq \bar{c}H_{\text{Ising}}(\gamma) \tag{3.36}$$

Note that this bound is independent of N and of the compatible family of contours $\{\Gamma_{\alpha(1)},\ldots,\Gamma_{\alpha(N)}\}$. To bound the term

$$\sum_{\substack{i\in I_+(\gamma)\\ j\in I_+(\Gamma_{\alpha(N)}\setminus\gamma)}} g(i-j)\,\mu_{\Gamma_{\alpha(1)},\ldots,\Gamma_{\alpha(N)}}(2(n_i-n_j)\,s_i-1)$$

in the exponent on the rhs of (3.33), we proceed as follows.

Using the fact that for $i\in I_+(\gamma)$, $j\in I_+(\Gamma_{\alpha(N)}\setminus\gamma)$, the expression $2(n_i-n_j)\,s_i$ can be negative only if $s_i=s_j$, we get the bound

$$\mu_{\Gamma_{\alpha(1)},\ldots,\Gamma_{\alpha(N)}}(2(n_i-n_j)\,s_i)\geqq -2\mu_{\Gamma_{\alpha(1)},\ldots,\Gamma_{\alpha(N)}}(|n_j|-2N) \tag{3.37}$$

From (3.34), (3.36), and (3.37), and the bound

$$(\bar{c}-c)\,H_{\text{Ising}}(\gamma)>\frac{1}{2}\sum_{\substack{i\in I_+(\gamma)\\ j\in I_+(\Gamma_{\alpha(N)}\setminus\gamma)}} g(i-j) \tag{3.38}$$

which holds for any $\bar{c}-c>0$ if M is sufficiently large, we conclude inequality (3.30). ▮

In our proof of the Peierls estimate we shall need a bound on the modulus of the rhs of (3.37) as N tends to ∞. This is provided by the next lemma.

Lemma 3.2. Let F be an increasing, exponentially bounded function on $\mathbb{Z}$ with

$$F(0)=0$$

and set

$$E_N(i_0) \equiv \mu_{\Gamma_{\alpha(1)},\dots,\Gamma_{\alpha(N)}}(F(n_{i_0}-2N)(\sigma_{i_0}(\Gamma_{\alpha(N)})+1)) \tag{3.39}$$

for any compatible sequence $\{\Gamma_{\alpha(1)},\dots,\Gamma_{\alpha(N)}\}$, $N\in\mathbb{N}$, and any $i_0\in\Lambda$. Then

$$\lim_{N\to\infty} E_N(i_0)=0 \tag{3.40}$$

Note that (3.40) is trivial unless $\sigma_{i_0}(\Gamma_{\alpha(N)})=+1$, i.e., $|n_{i_0}|\geqq 2N$.

This implies, in particular, that for any $h\in\mathbb{R}_+$ and any $i_0\in\Lambda$ with $\sigma_{i_0}(\Gamma_{\alpha(N)})=+1$

$$\lim_{N\to\infty} \mu_{\Gamma_{\alpha(1)},\dots,\Gamma_{\alpha(N)}}(e^{h(n_{i_0}-2N)}-1)=0 \tag{3.41}$$

and

$$\lim_{N\to\infty} \mu_{\Gamma_{\alpha(1)},\dots,\Gamma_{\alpha(N)}}(|n_{i_0}|-2N)=0 \tag{3.42}$$

Remark. In (3.42) we use the symmetry of the measure $\mu_{\Gamma_{\alpha(1)},\dots,\Gamma_{\alpha(N)}}$ in n.

Proof. Since the measures $\mu_{\Gamma_{\alpha(1)},\dots,\Gamma_{\alpha(N)}}$ fulfill FKG inequalities, we conclude, using the fact that the function

$$\prod_{i\in\Lambda} \chi(n_i\geqq 0)\equiv \chi_{\Lambda,\mathbb{N}} \tag{3.43}$$

[with $\chi(n_i\geqq 0)$ the characteristic function of the set $\{n_i\geqq 0\}$] is nondecreasing, that

$$\mu_{\Gamma_{\alpha(1)},\dots,\Gamma_{\alpha(N)}}(F)\leqq \frac{\mu_{\Gamma_{\alpha(1)},\dots,\Gamma_{\alpha(N)}}(\chi_{\Lambda,\mathbb{N}}\cdot F)}{\mu_{\Gamma_{\alpha(1)},\dots,\Gamma_{\alpha(N)}}(\chi_{\Lambda,\mathbb{N}})} \tag{3.44}$$

Next, applying FKG inequalities to the conditional expectations with respect to $\{n_j, \sigma_j(\Gamma_{\alpha(N)}) = -1\}, j \in \Lambda$, associated to the probability measure on the rhs of (3.44), we find that

$$\frac{\mu_{\Gamma_{\alpha(1)},\dots,\Gamma_{\alpha(N)}}(\chi_{\Lambda,\mathbb{N}} \cdot F(n_{i_0}))}{\mu_{\Gamma_{\alpha(1)},\dots,\Gamma_{\alpha(N)}}(\chi_{\Lambda,\mathbb{N}})} \leq \frac{\sum'_{n_i,\sigma_i(\Gamma_{\alpha(N)})=+1} e^{-\beta H(n)} F(n_{i_0} - 2N)}{\sum'_{n_i,\sigma_i(\Gamma_{\alpha(N)})=+1} e^{-\beta H(n)}} \tag{3.45}$$

where $\sum'$ only extends over configurations n for which $n_j = 2N$ if $\sigma_j(\Gamma_{\alpha(N)}) = -1$ for all j.

Changing the summation variables $n_i \to n_i + 2N$, we get that

$$\begin{aligned}\text{rhs}(3.45) = &\left[\sum_{\substack{n_i \in \mathbb{Z}^+: \\ \sigma_i(\Gamma_{\alpha(N)}) = +1}} (\exp\{-\beta[\tilde{H}(n+2N) - \tilde{H}(2N)]\})\, F(n_{i_0})\right] \\ &\times \left(\sum_{\substack{n_i \in \mathbb{Z}^+: \\ \sigma_i(\Gamma_{\alpha(N)}) = +1}} \exp\{-\beta[\tilde{H}(n+2N) - \tilde{H}(2N)]\}\right)^{-1}\end{aligned} \tag{3.46}$$

with

$$\begin{aligned}\tilde{H}(n+2N) = &\frac{1}{2} \sum_{\substack{j \in \Lambda^c \\ i \in \Lambda, \sigma_i(\Gamma_{\alpha(N)}) = +1}} g(i-j)(n_i + 2N)^2 \\ &+ \frac{1}{2} \sum_{\substack{j \in \Lambda, \sigma_j(\Gamma_{\alpha(N)}) = -1 \\ i \in \Lambda, \sigma_i(\Gamma_{\alpha(N)}) = +1}} g(i-j)\, n_i^2 \\ &+ \frac{1}{2} \sum_{\substack{i,j \in \Lambda \\ \sigma_i(\Gamma_{\alpha(N)}) = +1 = \sigma_j(\Gamma_{\alpha(N)})}} g(i-j)(n_i - n_j)^2\end{aligned} \tag{3.47}$$

Then, for the numerator on the rhs of (3.46), we get, using $F(0) = 0$, that

$$\begin{aligned}&\sum_{\substack{n_i \in \mathbb{Z}^+: \\ \sigma_i(\Gamma_{\alpha(N)}) = +1}} \exp\{-\beta[\tilde{H}(n+2N) - \tilde{H}(2N)]\}\, F(n_{i_0}) \\ &\quad \leq \exp\left\{-2N\beta\left[\sum_{j \in \Lambda^c} g(i_0 - j)\right]\right\} \\ &\quad\quad \times \sum_{n_i \in \mathbb{Z}^+ : i \in \Lambda} \{\exp[-\beta H(n)]\}\, F(n_{i_0})\end{aligned} \tag{3.48}$$

The denominator in (3.46) satisfies the obvious lower bound

$$\sum_{\substack{n_i \in \mathbb{Z}^+: \\ \sigma_i(\Gamma_{\alpha(N)}) = +1}} \exp\{-\beta[\tilde{H}(n+2N) - \tilde{H}(2N)]\} \geq 1 \tag{3.49}$$

Using that Λ is a bounded set, we conclude from (3.44)–(3.45) and (3.46)–(3.49) that

$$E_N(i_0) \leqq e^{-2N\beta a} C_\Lambda \tag{3.50}$$

where $E_N(i_0)$ has been defined in (3.39),

$$a \equiv \inf_{i \in \Lambda} \sum_{j \in \Lambda^c} g(i-j) > 0 \tag{3.51}$$

and

$$C_\Lambda \equiv \sup_{i_0 \in \Lambda} \left(\sum_{n_j \geqq 0} e^{-\beta H(n)} F(n_{i_0}) \right) \quad \blacksquare \tag{3.52}$$

Remark. Our proof shows that the rate of convergence in (3.40) is independent of the family $\{\Gamma_{\alpha(1)},..., \Gamma_{\alpha(N)}\}_{N \in \mathbb{N}}$. Moreover, the lemma also holds for adiabatic (Dirichlet) boundary conditions, as follows from the estimates shown above.

For a primitive contour γ we denote

$$P_{\Gamma_{\alpha(1)},..., \Gamma_{\alpha(N)}}(\gamma) \equiv \mu_{\Gamma_{\alpha(1)},..., \Gamma_{\alpha(N)}}\{n \in \Omega_\Lambda : \gamma \subset \Gamma, \chi_\Gamma^{(N+1)}(n) = 1\} \tag{3.53}$$

i.e., $P_{\Gamma_{\alpha(1)},..., \Gamma_{\alpha(N)}}(\gamma)$ is the probability, computed with the measure $\mu_{\Gamma_{\alpha(1)},..., \Gamma_{\alpha(N)}}$, for the primitive contour γ to appear in the configurations of n on level $(N+1)$. We now prove that $P_{\Gamma_{\alpha(1)},..., \Gamma_{\alpha(N)}}(\gamma)$ is bounded above by the probability of the event that the primitive contour γ appears in a spin configuration of the corresponding Ising model, provided the temperature is assumed to be sufficiently low. As one can expect from our previous considerations, we shall simultaneously show that $\mu_{\Gamma_{\alpha(1)},..., \Gamma_{\alpha(N)}}((|n_i| - 2N)(\sigma_i(\Gamma_{\alpha(N)}) + 1))$ is uniformly bounded in $i \in \Lambda$, Λ, and $\{\Gamma_{\alpha(1)},..., \Gamma_{\alpha(N)}\}$, $N \in \mathbb{N}$.

We now show that, at sufficiently low temperatures, the Peierls estimate can be completed on any level $N \in \mathbb{N}$.

Proposition 3.1. There is a finite constant $\beta_0 > 0$ such that, for any $\beta > \beta_0$ the following bounds hold:

(P1) For any $N \in \mathbb{N}$, $\{\Gamma_{\alpha(1)},..., \Gamma_{\alpha(N)}\}$, and an arbitrary primitive contour γ:

$$P_{\Gamma_{\alpha(1)},..., \Gamma_{\alpha(N)}}(\gamma) \leqq \exp\left\{-\beta \frac{c}{2} H_{\text{Ising}}(\gamma)\right\} \tag{3.54}$$

(P2) for any $i \in \Lambda$,

$$\mu_{\Gamma_{\alpha(1)},\ldots,\Gamma_{\alpha(N)}}((|n_i| - 2N)\, \tfrac{1}{2}(\sigma_i(\Gamma_{\alpha(N)}) + 1)) \leqq 2(1 + e^{-\beta a}) \tag{3.55}$$

for some constants a, $c > 0$ independent of $N \in \mathbb{N}$, $\{\Gamma_{\alpha(1)},\ldots, \Gamma_{\alpha(N)}\}$, γ, $i \in \Lambda$, Λ, and $\beta > \beta_0$.

Proof. The proof proceeds by induction. Take $N \equiv N(\Lambda)$ sufficiently large, so that, for any compatible family $\{\Gamma_{\alpha(1)},\ldots, \Gamma_{\alpha(N)}\}$, we have (from Lemma 3.2)

$$\mu_{\Gamma_{\alpha(1)},\ldots,\Gamma_{\alpha(\bar{N})}}[(|n_i| - 2\bar{N})\, \tfrac{1}{2}(\sigma_i(\Gamma_{\alpha(\bar{N})}) + 1)] \leqq 2 \tag{3.56}$$

Take $M > 1$ in condition (D) (Section 2.2) such that, for any primitive contour γ and a set Γ of spin flips with $\gamma \subseteq \Gamma$, we have that

$$\frac{c}{2} H_{\text{Ising}}(\gamma) - 3 \sum_{\substack{i \in I_+(\gamma) \\ j \in I_+(\Gamma \setminus \gamma)}} g(i-j) \geqq 0 \tag{3.57}$$

(From considerations of ref. 1 it follows that this is possible.)

We now prove the induction step. Assuming that (P2) has been shown on all levels $\geqslant N$, we prove (P1) and (P2) on level $N-1$.

From considerations similar to those leading to (3.6), we have that, for any compatible family $\{\Gamma_{\alpha(1)},\ldots, \Gamma_{\alpha(N)}\}$,

$$P_{\Gamma_{\alpha(1)},\ldots,\Gamma_{\alpha(N-1)}}(\gamma) \leqq \frac{\sum_{\Gamma:\, \gamma \subset \Gamma} \mu_{\Gamma_{\alpha(1)},\ldots,\Gamma_{\alpha(N-1)}} \chi_\Gamma^{(N)}}{\sum_{\Gamma:\, \gamma \subset \Gamma} \mu_{\Gamma_{\alpha(1)},\ldots,\Gamma_{\alpha(N-1)}} \chi_\Gamma^{(N)} (\mu_{\Gamma_{\alpha(1)},\ldots,\Gamma_{\alpha(N-1)}} \tilde{\chi}_\Gamma^{(N)} / \mu_{\Gamma_{\alpha(1)},\ldots,\Gamma_{\alpha(N-1)}} \chi_\Gamma^{(N)})} \tag{3.58}$$

Hence using Lemma 3.1 together with (3.55) and (3.57), we obtain the upper bound

$$P_{\Gamma_{\alpha(1)},\ldots,\Gamma_{\alpha(N-1)}}(\gamma) \leqq \exp\left\{-\beta \frac{c}{2} H_{\text{Ising}}(\gamma)\right\} \tag{3.59}$$

This completes the proof of (P1) on level $(N-1)$. To show (P2), we note next that, for any $i \in \Lambda$,

$$\begin{aligned} &\mu_{\Gamma_{\alpha(1)},\ldots,\Gamma_{\alpha(N-1)}}((|n_i| - 2(N-1))\, \tfrac{1}{2}(\sigma_i(\Gamma_{\alpha(N-1)}) + 1)) \\ &\quad \leqq 2 + \mu_{\Gamma_{\alpha(1)},\ldots,\Gamma_{\alpha(N-1)}}((|n_i| - 2N)\, \chi(|n_i| \geqq 2N)) \end{aligned} \tag{3.60}$$

The second term on the rhs of (3.60) can be written as follows:

$$\mu_{\Gamma_{\alpha(1)},\dots,\Gamma_{\alpha(N-1)}}[(|n_i|-2N)\,\chi(|n_i|\geqq 2N)]$$
$$=\sum_{\gamma:\,\sigma_i(\gamma)=+1}\;\sum_{\Gamma:\,\gamma\subset\Gamma}(\mu_{\Gamma_{\alpha(1)},\dots,\Gamma_{\alpha(N-1)}}\chi_{\Gamma}^{(N)})$$
$$\cdot\,\mu_{\Gamma_{\alpha(1)},\dots,\Gamma_{\alpha(N-1)},\Gamma}(|n_i|-2N) \tag{3.61}$$

Now, using our assumption (3.55), we arrive at

$$\mu_{\Gamma_{\alpha(1)},\dots,\Gamma_{\alpha(N-1)}}(|n_i|-2N)\,\chi(|n_i|\geqq 2N)$$
$$\leqq 3\sum_{\gamma:\,\sigma_i(\gamma)=+1}P_{\Gamma_{\alpha(1)},\dots,\Gamma_{\alpha(N-1)}}(\gamma) \tag{3.62}$$

Using (3.59) and the entropy estimate (2.16) for primitive contours, we conclude that, for $\beta>\beta_0$, where $\beta_0>0$ is chosen sufficiently large (independently of Λ, $\{\Gamma_{\alpha(1)},\dots,\Gamma_{\alpha(N-1)}\}$, and N),

$$\sum_{\gamma:\,\sigma_i(\gamma)=+1}P_{\Gamma_{\alpha(1)},\dots,\Gamma_{\alpha(N-1)}}(\gamma)\leqq e^{-\beta\tilde{a}} \tag{3.63}$$

where the constant $\tilde{a}>0$ depends only on $c>0$ [and on the constants C_1 and C_2 from (1.13)–(1.17)]. From (3.60) and (3.61)–(3.63) we obtain (P2) on level $(N-1)$, i.e.,

$$\mu_{\Gamma_{\alpha(1)},\dots,\Gamma_{\alpha(N-1)}}((|n_i|-2(N-1))(\tfrac{1}{2}(\sigma_i(\Gamma_{\alpha(N-1)}+1)))\leqq 2(1+e^{-\beta a}) \tag{3.64}$$

We can and do assume that $\beta_0>0$ is such that

$$e^{-\beta\tilde{a}}\leqq 1/3 \tag{3.65}$$

Note that, by Lemma 3.1 and (3.58), we have that (3.59) and hence (3.63) remain true for $K<\bar{N}$ and any $\{\Gamma_{\alpha(1)},\dots,\Gamma_{\alpha(K)}\}$, as long as

$$\mu_{\Gamma_{\alpha(1)},\dots,\Gamma_{\alpha(K+1)}}((|n_i|-2(K+1))\,\tfrac{1}{2}(\sigma_i(\Gamma_{\alpha(K+1)})+1))\leqq 3 \tag{3.66}$$

for any $i\in\Lambda$ [since (3.66) guarantees that we can use (3.57)].

Hence, in order to complete our induction, it suffices to note that we have the following flow for the expectations of primary interest to us:

$$\mu_{\Gamma_{\alpha(1)},\dots,\Gamma_{\alpha(K)}}((|n_i|-2K)\,\tfrac{1}{2}(\sigma_i(\Gamma_{\alpha(K)})+1))\leqq 2\left(\sum_{m=0}^{(\bar{N}-K)}e^{-\beta\tilde{a}m}\right) \tag{3.67}$$

This follows by arguments identical to those used in (3.60)–(3.64). Now we see that our choice (3.65) of β_0 implies that the rhs of (3.67) is bounded above by 3, for any $\bar{N}$ and K. This shows that our induction is independent of Λ if $\beta > \beta_0 > 0$ is sufficiently large. This ends the proof by taking suitable $a > 0$. ∎

3.3. Restricted Analyticity at Low Temperature

Suppose that β_0 is so large that Proposition 3.1 holds.

Proposition 3.2. For any $\beta > \beta_0$ there is some finite constant $\bar{h} \equiv \bar{h}(\beta) > 0$ such that for any $|h| < \bar{h}$

$$\mu_{\Lambda,\beta} e^{hn_i} < e^h \cdot [1 - e^{-(\beta C - 2\bar{h})}]^{-1} \tag{3.68}$$

for some constant $C > 0$ independent of $i \in \Lambda$ and Λ.

Proof. We follow the idea of the proof of (P2) and use Lemma 3.2, (3.41). For $i \in \Lambda$ and $h > 0$ let us note the simple fact that

$$\begin{aligned} \mu_{\Lambda,\beta} e^{hn_i} &\leqq e^h + e^{2h} \mu_{\Lambda,\beta}(e^{h(n_i-2)} \chi(|n_i| \geqq 2)) \\ &= e^h + e^{2h} \sum_{\gamma:\, \sigma_i(\gamma) = +1} \left(\sum_{\Gamma:\, \gamma \subset \Gamma} (\mu_{\Lambda,\beta} \chi_\Gamma^{(1)})\, \mu_\Gamma e^{h(n_i-2)} \right) \end{aligned} \tag{3.69}$$

Repeating the same arguments for $\mu_\Gamma e^{h(n_i-2)}$ and inserting the result on the rhs of (3.69), we get that

$$\begin{aligned} \mu_{\Lambda,\beta} e^{hn_i} \leqq e^h &+ e^{2h} \sum_{\gamma_1:\, \sigma_i(\gamma_1) = +1} \left(\sum_{\Gamma_1:\, \gamma_1 \subset \Gamma_1} \mu_{\Lambda,\beta} \chi_{\Gamma_1}^{(1)} \right) e^h \\ &+ e^{2h} \sum_{\gamma_1:\, \sigma_i(\gamma_1) = +1} \left[\sum_{\Gamma_1:\, \gamma_1 \subset \Gamma_1} \mu_{\Lambda,\beta}(\chi_{\Gamma_1}^{(1)}) \right. \\ &\left. \times e^{2h} \sum_{\gamma_2:\, \sigma_i(\gamma_2) = +1} \left(\sum_{\Gamma_2:\, \gamma_2 \subset \Gamma_2} \mu_{\Gamma_1,\Gamma_2} e^{h(n_i-4)} \right) \right] \end{aligned} \tag{3.70}$$

This bound can be rewritten as follows:

$$\begin{aligned} \mu_{\Lambda,\beta} e^{hn_i} \leqq e^h &\left[1 + e^{2h} \sum_{\gamma_1:\, \sigma_i(\gamma_1) = +1} P_0(\gamma_1) \right] \\ &+ e^{2h} \sum_{\gamma_1:\, \sigma_i(\gamma_1) = +1} \left(\mu_{\Lambda,\beta} \chi_{\Gamma_1}^{(1)} e^{2h} \sum_{\gamma_2:\, \sigma_i(\gamma_2) = +1} \sum_{\Gamma_2:\, \gamma_2 \subset \Gamma_2} \mu_{\Gamma_1,\Gamma_2} e^{h(n_i-4)} \right) \end{aligned} \tag{3.71}$$

with $P_0(\gamma_1)$ the probability for a primitive contour γ_1 to appear, computed in the measure $\mu_{\Lambda,\beta} \equiv \mu_0$. By induction, using Lemma 3.2 with $N = N(\Lambda)$ chosen such that

$$\mu_{\Gamma_1,\dots,\Gamma_N} e^{h(n_i - 2N)} \leqq e^h \tag{3.72}$$

we get the bound

$$\mu_{\Lambda,\beta} e^{hn_i} \leqq e^h \Bigg\{ 1 + e^{2h} \sum_{\gamma_1:\, \sigma_i(\gamma_1) = +1} P_0(\gamma_1)$$
$$+ e^{2h} \sum_{\gamma_1:\, \sigma_i(\gamma_1) = +1} \mu_0(\chi^{(1)}_{\Gamma_1}) \left[e^{2h} \sum_{\gamma_2:\, \sigma_i(\gamma_2) = +1} P_{\Gamma_1}(\gamma_2) \right]$$
$$+ \cdots + e^{2h} \sum_{\gamma_1:\, \sigma_i(\gamma_1) = +1} \mu_0(\chi^{(1)}_{\Gamma_1}) \Bigg(e^{2h} \sum_{\gamma_2:\, \sigma_i(\gamma_2) = +1} \; \sum_{\Gamma_2:\, \gamma_2 \subset \Gamma_2} \mu_{\Gamma_1} \chi^{(2)}_{\Gamma_2}$$
$$\times \cdots e^{2h} \sum_{\gamma_{N-1}:\, \sigma_i(\gamma_{N-1}) = +1} \; \sum_{\Gamma_{N-1}:\, \gamma_{N-1} \subset \Gamma_{N-1}} \mu_{\Gamma_1,\ldots,\Gamma_{N-2}} \chi^{(N-1)}_{\Gamma_{N-1}}$$
$$\times e^{2h} \sum_{\gamma_N:\, \sigma_i(\gamma_N) = +1} P_{\Gamma_1,\ldots,\Gamma_{N-1}}(\gamma_N) \Bigg) \Bigg\} \tag{3.73}$$

Since from Proposition 3.1 (P1) and the entropy estimate (2.16) we conclude that

$$\sum_{\gamma_K:\, \sigma_i(\gamma_K) = +1} P_{\Gamma_1,\ldots,\Gamma_{K-1}}(\gamma_K) \leqq e^{-\beta C} \tag{3.74}$$

for a constant $C > 0$ independent of β, Λ, $i \in \Lambda$ and $\{\Gamma_1,\ldots,\Gamma_N\}$, we see that (3.73) implies the bound

$$\mu_{\Lambda,\beta} e^{hn_i} \leqq e^h \sum_{m=0}^{N(\Lambda)} (e^{2h} e^{-\beta C})^m \tag{3.75}$$

Hence, if

$$0 \leq h < \bar{h} \leqq \frac{\beta C}{2} \tag{3.76}$$

then

$$\mu_{\Lambda,\beta} e^{hn_i} \leqq e^h [1 - e^{-(\beta C - 2h)}]^{-1} \tag{3.77}$$

This completes the proof. ∎

Proposition 3.1 suggests that the Gaussian behavior of moments of $\mu_{\Lambda,\beta}$ is violated, since we have only the following bound:

$$\mu_{\Lambda,\beta}(n_i^{2r}) \leqq (2r)!\, (A(\beta))^r, \qquad r \in \mathbb{N} \tag{3.78}$$

with $0 < A(\beta) < \infty$ independent of Λ, $i \in \Lambda$, and $r \in \mathbb{N}$.

4. ROUGHENING IN THE DG MODEL AT HIGH TEMPERATURE

In this section we study the high-temperature phase of the discrete Gaussian chain with $1/r^2$ interaction energy. We propose to prove the bound (1.16), i.e., that there exists some positive, finite constant β_0 such that, for $\beta < \beta_0$,

$$\langle (n_i - n_j)^2 \rangle_\beta \geqq \text{const}_\beta \cdot \log |i-j| \tag{4.1}$$

as $|i-j| \to \infty$. We shall also prove that

$$\langle (n_i - n_j)^2 \rangle_\beta \leqq \text{const}'_\beta \cdot \log |i-j| \tag{4.2}$$

as $|i-j| \to \infty$, for all $\beta > 0$.

Inequality (4.1) shows that, at high temperatures, the interface described by the DG chain is rough. In Section 3 we have shown that, at sufficiently low temperatures and *for zero boundary conditions*,

$$\langle (n_i - n_j)^2 \rangle_{\beta, \Lambda} \leqq \text{const}''_\beta \tag{4.3}$$

uniformly in i, j, and $\Lambda = [-L, L]$. This follows from our bounds on $\langle n_j^2 \rangle_{\beta, \Lambda}$, i.e.,

$$\langle n_j^2 \rangle_{\beta, \Lambda} \leqq \text{const}'''_\beta \tag{4.4}$$

uniformly in Λ.

In order to complete our proof of the existence of a phase transition, we must prove (4.1) for the *same* choice of boundary conditions for which (4.3) and (4.4) were proven, i.e., for zero boundary conditions. Then (4.1) and (4.3) imply that the constant β_0 is strictly positive and finite.

Our proof of inequality (4.1) (for zero boundary conditions) is based on correlation inequalities, reviewed in Section 4.1, and a result due to Kjaer and Hilhorst[3]; see Section 4.2.

4.1. Some Useful Inequalities

Consider the DG chain on the interval $\Lambda = [-L, L]$ with Hamiltonian

$$H_g = \frac{1}{2} \sum_{i, j \in \Lambda} n_i g(i, j) n_j \tag{4.5}$$

where n_i, $i \in \Lambda$, is a real random variable with *a priori* distribution

$$d\rho_\lambda(n_i) = \exp \lambda \cos(2\pi n_i) \, dn_i \tag{4.6}$$

for some $\lambda \in (0, \infty]$. The Gibbs state of the system confined to Λ is given by

$$\langle F \rangle_{g,\lambda} \equiv \langle F \rangle_{g,\lambda,\Lambda} := Z_{g,\lambda,\Lambda}^{-1} \int e^{-H_g(n)} F(n) \prod_{i=-L}^{L} d\rho_\lambda(n_i) \tag{4.7}$$

where the partition function $Z_{g,\lambda,\Lambda}$ is chosen such that $\langle 1 \rangle_{g,\lambda} = 1$.

Let $f = (f(i))_{i \in \Lambda}$ and $h = (h(i))_{i \in \Lambda}$ be two sequences of complex numbers. We define their scalar product (f, h) by setting

$$(f, h) := \sum_{i \in \Lambda} \overline{f(i)}\, h(i)$$

This equips $\mathbb{C}^{2L+1}$ with a scalar product, hence making it a Hilbert space, which is commonly denoted by $l_2(\Lambda)$. The matrix $g(i, j)$ defines a quadratic form g on $l_2(\Lambda)$ by

$$(f, gh) := \sum_{i,j \in \Lambda} \overline{f(i)}\, g(i, j)\, h(j)$$

We say that $g_1 \geqq g_2$ iff

$$(f, g_1 f) \geqq (f, g_2 f) \qquad \text{for all} \quad f \in l_2(\Lambda) \tag{4.8}$$

We are now prepared to state some basic inequalities (of Ginibre type) proven in ref. 4: Let f be an arbitrary element of $l_2(\Lambda)$, and set $n(f) \equiv (n, f) = \sum_{j \in \Lambda} n_j f(j)$. Then

$$\langle |n(f)|^2 \rangle_{g_1,\lambda_1} \leqq \langle |n(f)|^2 \rangle_{g_2,\lambda_2} \tag{4.9}$$

whenever $g_1 \geqq g_2 > 0$ and $\lambda_1 \geqq \lambda_2$, i.e., $\langle |n(f)|^2 \rangle_{g,\lambda}$ is *monotone decreasing* in g and λ. As corrollaries of (4.9) we have that

$$\begin{aligned} \langle (n_i - n_j)^2 \rangle_g &\leqq \langle (n_i - n_j)^2 \rangle_{g,0} \\ &= (g^{-1})(i, i) + (g^{-1})(j, j) - 2(g^{-1})(i, j) \end{aligned} \tag{4.10}$$

$$\langle n_i^2 \rangle_g \leqq (g^{-1})(i, i) \tag{4.11}$$

where $\langle (\cdot) \rangle_g = \lim_{\lambda \nearrow \infty} \langle (\cdot) \rangle_{g,\lambda}$ is the expectation of the DG chain with interactions $g(i, j)$, g^{-1} is the inverse matrix of $g > 0$, and $\langle (\cdot) \rangle_{g,0}$ is the Gaussian expectation with mean 0 and covariance g^{-1}, as is seen from (4.7). Furthermore, we have that

$$\langle (n_i - n_j)^2 \rangle_{\beta g,\lambda} \text{ and } \langle n_i^2 \rangle_{\beta g,\lambda} \text{ are monotone decreasing in } \beta \geq 0 \tag{4.12}$$

for all $0 \leq \lambda \leq \infty$, whenever $g > 0$.

4.2. Consequences of the Results of Kjaer and Hilhorst[3]

By combining the so-called sine-Gordon transformation (see ref. 9 and references given there) with a transformation due to Cardy,[5] Kjaer and Hilhorst[3] have been able to analyze a special DG chain, at a special value of the inverse temperature β, explicitly. They choose periodic boundary conditions $n_{L+1+i} = n_{-L-1+i}$ and *pin* n_{L+1} *at height* 0, i.e., $n_{L+1} = 0$. Furthermore, they choose a self-dual interaction, $g(i, j) := g_L^*(i-j)$, given by

$$g_L^*(j) := \left[\frac{\pi}{2L+2} \sin\left(\frac{\pi}{2L+2}\right)\right] \times \left[\sin\left(\frac{\pi(j+\frac{1}{2})}{2L+2}\right) \sin\left(\frac{\pi(j-\frac{1}{2})}{2L+2}\right)\right]^{-1} \tag{4.13}$$

for $|j| = 1, \ldots, 2L-1$. By using a Fourier transform, it is not hard to show that $g_L^* > 0$, as a quadratic form, see (4.8); and from (4.13) one has that

$$g_L^*(j) \to g^*(j) = (j^2 - \tfrac{1}{4})^{-1}, \qquad j \in \mathbb{Z} \tag{4.14}$$

as $L \to \infty$.

It is then shown in ref. 3 that

$$\langle (n_i - n_j)^2 \rangle_{g^*} = \lim_{L \to \infty} \langle (n_i - n_j)^2 \rangle_{g_L^*} \simeq \frac{1}{2\pi^2} \log |i-j| \tag{4.15}$$

as $|i-j| \to \infty$.

Equation (4.15) follows quite easily from the self-duality of the model with $g = g_L^*$ and $\lambda = \infty$; see ref. 3.

The function g_L^* determines a positive-definite matrix $\tilde{g}_L = (\tilde{g}_L(i, j))$ such that, for an arbitrary sequence $f \in l_2(\Lambda)$, with $f(L+1) = 0$,

$$\sum_{-L \le i, j \le L+1} g_L^*(i-j) \, |f(i) - f(j)|^2 \Big|_{f(L+1)=0} = \sum_{-L \le i, j \le L} \overline{f(i)} \, \tilde{g}_L(i, j) \, f(j) \tag{4.16}$$

Let g_L be any matrix with the property that

$$0 < g_L \leqq A \tilde{g}_L \tag{4.17}$$

for some finite constant A. Let $0<\beta\leq A^{-1}$, so that $0<\beta g_L\leqq \tilde{g}_L$. Then it follows from inequality (4.9) that

$$\langle (n_i-n_j)^2\rangle_{\beta g_L}\geq \langle (n_i-n_j)^2\rangle_{\tilde{g}_L} \tag{4.18}$$

for *all* $0<\beta\leqq A^{-1}$.

Next, we observe that any function g on $\mathbb{Z}$ determines a quadratic form g_L on $l_2(\Lambda)$ by setting

$$\sum_{-L\leq i,j\leq L} \overline{f(i)}\, g_L(i,j)\, f(j) := \sum_{i,j\in\mathbb{Z}} g(i-j)\,|f(i)-f(j)|^2 \tag{4.19}$$

with $f(i)=0$, for all $i\in\mathbb{Z}\backslash\Lambda$.

The point is now to choose a function g with the properties that $0<g(j)\sim \text{const}\cdot |j|^{-2}$, as $|j|\to\infty$, that the estimates proven in Section 3 are valid for the DG chain with equilibrium state $\langle(\cdot)\rangle_{\beta g_L}$, g_L as in (4.19), for β and L large enough, and that

$$g_L\leq A\tilde{g}_L \tag{4.20}$$

for some $A<\infty$ independent of L, for L large enough. Then it follows from (4.18) and (4.15) that, for $0<\beta\leqq A^{-1}$,

$$\begin{aligned}\langle (n_i-n_j)^2\rangle_{\beta g_L} &\geqq \langle (n_i-n_j)^2\rangle_{g_L^*}\\ &\simeq \frac{1}{2\pi^2}\log|i-j|\end{aligned} \tag{4.21}$$

for L large enough, which proves (4.1).

It also follows from (4.10) that

$$\begin{aligned}\langle (n_i-n_j)^2\rangle_{\beta g_L} &\leqq \langle (n_i-n_j)^2\rangle_{\beta g_L,\lambda=0}\\ &\leqq \text{const}\cdot\beta^{-1}\log(|i-j|+1)\end{aligned} \tag{4.22}$$

This upper bound shows that the lower bound (4.21) is poor, for very small β, since the rhs of (4.21) is independent of β. This unsatisfactory state of affairs can be improved if one is willing to use the rather involved techniques developed in ref. 9. As shown in ref. 3, the DG chain at inverse temperature β is equivalent to a classical lattice gas of charges $q\in\mathbb{Z}$ interacting through a *logarithmic* potential (as $L\to\infty$), at inverse temperature $\beta^* := \beta^{-1}$. This lattice gas can be reconstructed from a *two-dimensional lattice Coulomb gas* of charges by "dimensional reduction," i.e., by confining the charges in the *two-dimensional* system to a line of sites $\{j=(j^1,j^2)\in\mathbb{Z}^2: j^2=0\}$. For β^* large enough, this system can be studied with the help of the techniques developed in ref. 9. These techniques prove that, for sufficiently large β^*, i.e., sufficiently small β,

$$\langle (n_i - n_j)^2 \rangle_{\beta g_L} \geqq \text{const}' \cdot \beta^{-1} \log |i-j| \tag{4.23}$$

for a suitable choice of g and L large enough. [In particular, thanks to inequality (4.9), the function g used in Section 3 and zero boundary conditions are compatible with the requirements of ref. 9.]

We note that the lower bound (4.23) has the same structure as the upper bound (4.22) (same β dependence, for small β), which is nice.

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